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REGULARIZATION OF INVERSE PROBLEMS BY THE SCHEME OF PARTIAL MATCHING WITH ELEMENTS OF A SET OF OBSERVATIONS

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The problem of determining the thermophysical properties by means of a discrete set of observations on the temperatures of the test object given with measurement errors is examined.

The investigation of complex processes by using inverse problems has attracted considerable attention lately. Their solution is associated with certain singularities, particularly the influence of errors in the initial data on the desired solution. As is known from [1], in such cases it is necessary to limit the domain of the allowable solutions and to match the measurement errors. Since a number of stabilizing functionals with the same problem can be set in correspondence and different norms for the deviation from the quantities observed can be selected, then it is interesting to determine those among them which will permit, for sufficiently general assumptions about the desired quantities, obtaining the most exact solutions under conditions of unimprovable observations for a broad range of measurement errors. In addition, the question of selecting the method of matching the observations occurs in the solution of applied ill-posed problems. One condition that establishes a relation between the accuracy of the solution and the measurement error [2] is used in the widespread problem, in practice, of restoring the thermal flux. This condition expresses the total error in all observations for measurements executed at several points. However, one condition can turn out to be inadequate to determine several parameters of a model that is characteristic for the inverse coefficient problems, while taking total account of the errors results in a loss in accuracy of the solution of the inverse problem [3]. This paper is devoted to investigating the properties of the regularized solution of an inverse coefficient problem for the nonlinear heat-conduction equation as a function of the degree of limitation of the domain of admissible solutions, the form of the observation error estimate, and the methods of matching them.

In the domain $Q = \{(x, t): 0 < x < 1, 0 < t < T\}$ we examine the one-dimensional heat-conduction equation

$$a_1 \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a_2 \frac{\partial u}{\partial x}\right) + f(x, t) \tag{1}$$

for which the initial and boundary conditions assuring uniqueness and stability in the determination of the function u(x, t) for given values of the specific heat $a_1(u)$ and the heat conductivity $a_2(u)$ and any T > 0 are assumed known.

Let us also assume that at m points of space, and for each of n times of the domain Q observation results are given

$$u_{ij}^{\delta} = \bar{u}(x_i, t_j) + \varepsilon_{ij}, \quad i = \overline{1, m}, \; j = \overline{1, n}, \tag{2}$$

with a known magnitude of the deviation norm

$$\delta_i^2 = \sum_{j=1}^{n} (u_{ij}^{\delta} - \bar{u}_{ij})^2, \quad i = \overline{1, m}, \tag{3}$$

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or

$$\Delta_i = \max_{j\in[1,n]} |u^{\delta}_{ij} - \bar{u}_{ij}|, \quad i = \overline{1, m}, \tag{4}$$

where $\bar{u}(x, t)$ is the temperature field of a rod with the thermophysical properties $\bar{a}_{1,2}(u)$, δ_i and Δ_i are values of the measurement error at the points of observation $\{x_i\}_{i=\overline{1,m}*}$.

Error estimates in the form of (3) or (4) are the most widespread forms of measurement error estimates and can be obtained by known methods of statistical processing of the results of experiment.

Let us pose the inverse coefficient problem for equation (1) to determine the functions $a_{1,2}(u)$ from a given set of observations u^{δ}, that has the prototype $\bar{u}$ satisfying (1). We use Tikhonov regularization by the particular matching scheme proposed in [3] for the solution. According to this scheme, the solution desired is determined in a domain which should be bounded by a certain stabilizing functional $\Omega[a]$, when matching the solution of the direct problem in each element $i = \overline{1,m}$ of the set of observations u^{δ}. In this case, the assignment of measurement interference distribution laws is not required, but both quantitative and qualitative information about the properties of the desired quantities can be involved in addition, e.g., in the form of giving monotonicity convexity sections and degrees of smoothness. Assignment of the observations in a whole series of points, for which the number needed at each point of observation x_i rather than by the total error is a distinctive feature of such a regularization scheme from those proposed earlier, and permits improvement of the accuracy of the solution of the inverse problem [3].

Let us examine the following stabilizing functionals for (1):

$$\Omega^{(k)}_{p,q}[a] = \begin{cases} \iint\limits_Q \left(\left[\dfrac{d^p a_1}{du^p}\right)^2 + \left(\dfrac{d^q a_2}{du^q}\right)^2\right] dxdt, & k = 1, \\ \iint\limits_Q \sum\limits_{i=1,2} \left[\left(\dfrac{\partial^p a_i}{\partial x^p}\right)^2 + \left(\dfrac{\partial^q a_i}{\partial t^q}\right)^2\right] dxdt, & k = 2, \\ \int\limits_{u_0}^{u_N} \left[\left(\dfrac{d^p a_1}{du^p}\right)^2 + \left(\dfrac{d^q a_2}{du^q}\right)^2\right] du, & k = 3, \end{cases} \tag{5}$$

where p and q are the orders of the stabilizer; k, its class; $u_0 = \min\limits_{ij} u^{\delta}_{ij}$ and $u_N = \max\limits_{i,j} u^{\delta}_{ij}$, limits of the segment of approximation. For a methodological study of the stabilizers introduced, it is assumed that $\operatorname{var}\limits_{x,t\in Q} u(x, t) \in [u_0, u_N]$. Let us note that the stabilizer properties were considered earlier in [4], but for another regularization scheme.

The order of the stabilizer is selected with the form of the desired functions $a_{1,2}(u)$ taken into account, which is determined by the composition of the a priori information about the object of investigation. In the absence of a sufficient quantity of such information, but under the assumption of the differentiability of $a_{1,2}(u)$, cubic splines can be recommended [5]:

$$a_1^{(l)}(u) = \lambda_{4l-3}S_1^{(l)} + \lambda_{4l-2}S_2^{(l)} + \lambda_{4l+1}S_3^{(l)} + \lambda_{4l+2}S_4^{(l)}, \quad u \in [u_{l-1}, u_l],$$

$$a_2^{(l)}(u) = \lambda_{4l-1}S_1^{(l)} + \lambda_{4l}S_2^{(l)} + \lambda_{4l+3}S_3^{(l)} + \lambda_{4l+4}S_4^{(l)}, \quad u \in [u_{l-1}, u_l],$$

where

$$S_1^{(l)} = \frac{(u_l - u)^2\,[2(u - u_{l-1}) + h_l]}{h_l^3}; \quad S_2^{(l)} = \frac{(u_l - u)^2\,(u - u_{l-1})}{h_l^2};$$

$$S_3^{(l)} = \frac{(u - u_{l-1})^2\,[2(u_l - u) + h_l]}{h_l^3}; \quad S_4^{(l)} = \frac{(u - u_{l-1})^2\,(u - u_l)}{h_l^2};$$

$h_l = u_l - u_{l-1}$; $l = \overline{1, N}$; N is a parameter of the approximation lattice, $\{u_l\}_{l=\overline{1,N}}$; $\{\lambda_i\}_{i=\overline{1,4(N+1)}}$ are the spline coefficients. Use of splines permits the introduction of an irregular approximation mesh, provides a sufficient degree of smoothness together with locality of the approximation and allows for the possibility of using up to third-order stabilizers, inclusive.

Reducing the stabilizing functional (5) to functions of the variable λ_i, we obtain the following quadratic form:

$$\Omega_{p,q}^{(k)}[\lambda] = \sum_{l=1}^{N} \Big\{ \sum_{i=1,2} \sum_{j=1,2} [\lambda_{i+4(l-1)}\, \omega_{i,j}^{p,l}\, \lambda_{j+4(l-1)} + \lambda_{i+4l}\, \omega_{i+2,j}^{p,l}\, \lambda_{j+4l}] +$$

$$+ \sum_{i=3,4} \sum_{j=3,4} [\lambda_{i+4(l-1)}\, \omega_{i,j}^{q,l}\, \lambda_{j+4(l-1)} + \lambda_{i+4l}\, \omega_{i-2,j}^{q,l}\, \lambda_{j+4l}] \Big\},$$

where

$$\omega_{i,j}^{p,l} = \begin{cases} \displaystyle\iint_{Q_l} \frac{d^p S_i^{(l)}}{dv^p} \frac{d^p S_j^{(l)}}{dv^p}\, d\xi d\tau, \quad k = 1; \\ \displaystyle\iint_{Q_l} \Big\{ \frac{dS_i^{(l)}}{dv} \frac{dS_j^{(l)}}{dv} \Big[\Big(\frac{\partial^p v}{\partial \xi^p}\Big)^2 + \Big(\frac{\partial^p v}{\partial \tau^p}\Big)^2 \Big] + \\ \displaystyle + 2(p-1) \frac{d^2 S_i^{(l)}}{dv^2} \frac{dS_j^{(l)}}{dv} \Big[\Big(\frac{\partial v}{\partial \xi}\Big)^2 \frac{\partial^2 v}{\partial \xi^2} + \Big(\frac{\partial v}{\partial \tau}\Big)^2 \times \\ \displaystyle \times \frac{\partial^2 v}{\partial \tau^2} \Big] + (p-1) \frac{d^2 S_i^{(l)}}{dv^2} \frac{d^2 S_j^{(l)}}{dv^2} \Big[\Big(\frac{\partial v}{\partial \xi}\Big)^4 + \\ \displaystyle + \Big(\frac{\partial v}{\partial \tau}\Big)^4 \Big] \Big\}\, d\xi d\tau, \quad k = 2; \\ \displaystyle\int_{u_{l-1}}^{u_l} \frac{d^p S_i^{(l)}}{dv^p} \frac{d^p S_j^{(l)}}{dv^p}\, dv, \quad k = 3; \end{cases}$$

$$Q_l = \{(\xi, \tau) : u_{l-1} \leqslant v(\xi, \tau) \leqslant u_l\}.$$

According to the regularization scheme selected, the solution of the inverse problem for the quadratic form $\Omega_{p,q}^{(k)}|\lambda|$ is represented as

$$\min_{\lambda \in E^{4(N+1)}} \Omega_{p,q}^{(k)}[\lambda] \tag{6}$$

under the conditions

$$\sum_{j=1}^{n} (u_{ij}^{\delta} - u_{ij})^2 \leqslant \delta_i^2, \quad i = \overline{1, m}, \tag{7}$$

or

$$\max_{j \in [1,n]} |u_{ij}^{\delta} - u_{ij}| \leqslant \Delta_i, \quad i = \overline{1, m}, \tag{8}$$

where $u_{ij} = u(x_i, t_j)$ are values of the solution of the direct problem for (1) at given points of observation for the selected spline coefficients.

The case when the class of functions to which the desired quantities belong can be broader, and the class of functions to which the selected cubic splines belong can be narrower, should be kept in mind in analyzing the properties of the solutions of the inverse problems. Taking this into account, we consider identification of the thermophysical properties described by polynomials with powers less than as well as greater than three.

To solve the model problem, we model the sample u^{δ} according to (2) at two points of observation $x_1 = 1/3$ and $x_2 = 2/3$ ($m = 2$) with the number of time measurements $n = 10$. We give the interference in the measurements by a random number transducer with a normal distribu-

TABLE 1. Identification of Linear Laws of Thermophysical Property Variation for $\sigma^2 = 10^{-4}$

$\|u_i^\delta - u_i\|$	Stabilizer	$\Omega_{0,0}^{(1)}$	$\Omega_{1,1}^{(1)}$	$\Omega_{2,2}^{(1)}$	$\Omega_{3,3}^{(1)}$	$\Omega_{1,1}^{(2)}$	$\Omega_{2,2}^{(2)}$
$\left[\sum_{j=1}^{n}(u_{ij}^\delta - u_{ij})^2\right]^{\frac{1}{2}}$	$\int_{u_0}^{u_N}(a_1^\delta - \bar{a}_1)^2 du$	0,0982	0,00558	$7{,}4\cdot10^{-4}$	$1{,}2\cdot10^{-4}$	0,0194	0,00731
	$\int_{u_0}^{u_N}(a_2^\delta - \bar{a}_2)^2 du$	0,1035	0,00087	$9{,}3\cdot10^{-4}$	$2{,}7\cdot10^{-5}$	0,0072	0,00089
$\max_{j\in[1,n]}\|u_{ij}^\delta - u_{ij}\|$	$\int_{u_0}^{u_N}(a_1^\delta - \bar{a}_1)^2 du$	0,0602	0,00403	$0{,}51\cdot10^{-7}$	$0{,}27\cdot10^{-7}$	$0{,}98\cdot10^{-4}$	$3{,}1\cdot10^{-6}$
	$\int_{u_0}^{u_N}(a_2^\delta - \bar{a}_2)^2 du$	0,0875	0,00207	$0{,}14\cdot10^{-7}$	$0{,}09\cdot10^{-7}$	0,00249	$2{,}65\cdot10^{-6}$

tion law and zero mathematical expectation in the function $\bar{u}(x, t)$, which is a solution of (1) with the homogeneous boundary conditions

$$\bar{u}|_{t=0} = 0, \quad \bar{u}|_{x=0} = 0, \quad \bar{u}|_{x=1} = 0 \tag{9}$$

and the selection of $\bar{a}_1 = 1 + u$, $\bar{a}_2 = (1 + u)/2$, f = 10, T = 1 as well as $\bar{a}_1 = 1 + u - u^3 + u^5/4$, $\bar{a}_2 = 1 + 2u - 3u^3 + u^5$, f = 50, T = 0.1. For given values of the functions $a_{1,2}$ and f, the problem (1) and (9) was solved by an implicity iteration finite-difference scheme [6]. The function u(x, t) was found by interpolation by the Bessel formula in the mesh function with the nodes $w_{x \times t} = 50 \times 100$, and the integration in calculating the stabilizers was by the Simpson cubature formula [7]. The mathematical programming problems (6) and (7) and (6) and (8) were reduced by the penalty function method [8] to absolute minimization problems for whose solution coordinate by coordinate descent was used. The penalty function was selected in the form

$$F(\lambda) = \Omega_{p,q}^{(k)}[\lambda] + \sum_{i=1}^{m} K_i r_i(\lambda),$$

where K_i are the penalty coefficients, $r_i(\lambda)$ is the residual by conditions (7) or (8). Its minimization was realized according to the scheme

$$\lambda_i^{(s+1)} = \begin{cases} \lambda_i^{(s)} + \mu_i^{(s)}, & F^{(s)} - F^{(s-1)} \leqslant 0, \\ \lambda_i^{(s)} - \mu_i^{(s)}, & F^{(s)} - F^{(s-1)} > 0, \end{cases}$$

where $\mu_i^{(s)}$ are the search scales for the s-th step which diminish under the condition of two unsuccessful attempts in succession, and increase otherwise.

Selection of the lattice parameter N is quite important in constructing an approximation by cubic splines. In real processes the desired dependences can be represented by complex functional laws whose nature is not known in advance. In this connection, the variations in the parameter N are dictated by the results of preliminary computations. Analysis of the values found here for the desired quantities and the residuals r_i by the matching conditions can clarify the nature of the solution found, and therefore, indicate the further behavior of the parameter N. In the example under consideration, the parameter N = 5 was selected as the preliminary value.

The results of identification of the thermophysical properties in the temperature segment $u \in [u_0, u_N]$ show that the accuracy of the solution of the inverse problem depends on the form of the matching with the errors in the experimental data, and is also associated with the form of the stabilizing functional. Utilizing the error estimate mode in form (8) as compared to mode (7) permits finding a more uniform approximation to the true values (Table 1). Matching by the cubic norm (8) also results in satisfactory identification in the case of an increase in the variance of the interference (Table 2). However, it should here be kept in mind that the space defined by the cubic norm is not strictly normalized [7]; hence, the appearance of local minimums is possible in the solution of the problem (6) and (8).

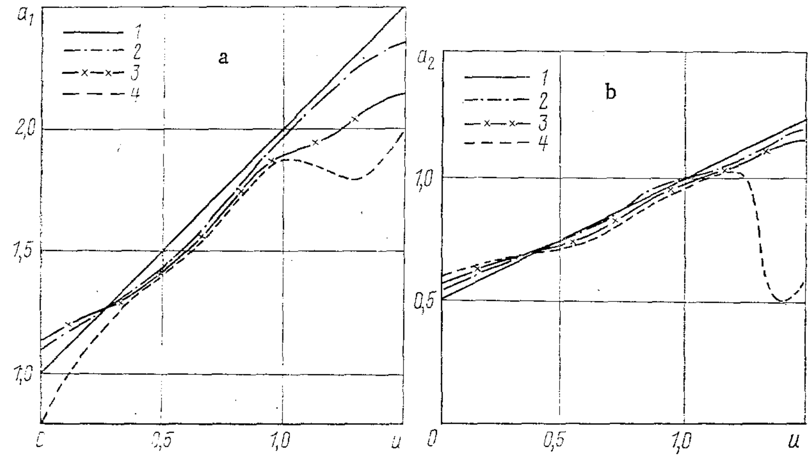

Fig. 1. Determination of linear laws of specific heat (a) and heat conductivity (b) variation for $\sigma = 0.01$ and matching by the spherical norm (7); 1) exact value; 2) $\Omega^{(2)}_{2,2}$; 3) $\Omega^{(2)}_{1,1}$; 4) $\Omega^{(1)}_{0,0}$.

In all the cases considered, the solutions $a^{\delta}_{1,2}$, found by using the numerical methods elucidated above, satisfy the necessary condition for minimizing the stabilizing functional $\Omega[a^{\delta}] < \Omega[\bar{a}]$, as well as a matching condition of the form (7) and (8), whereupon $r_i < 0$, $i = \overline{1, m}$. Since an adequate model and exact values of the errors found by (3) and (4) were used in solving the inverse problem, the result of negative residuals then obtained in conditions (7) and (8) shows that in similar situations the matching conditions can reduce to the requirement of satisfying the equalities $r_i = 0$, $i = \overline{1, m}$.

The functionals (5) introduced are distinguished by the degree of constraint imposed on the domain of allowable solutions. Consequently, the functions $a^{\delta}_{1,2}$ found for the different stabilizers but under the same matching conditions are distinctive in the accuracy of their approximation to the true values $\bar{a}_{1,2}$. The accuracy of the identification is improved with the rise in the stabilizer order to $p = q = 2$. A further increase in the order does not, in some cases, result in improvement in the accuracy of identification. The thermophysical properties found are shown in Figs. 1 and 2 as a function of the form of the stabilizing function used. Let us note that the results obtained on the basis of using first- and second-order stabilizers differ significantly from each other. This difference is related to the kind of approximation used for the desired quantities. Since cubic splines belong to the class of twice differentiable functions, then the constraint on the domain of admissible solutions on the basis of the first derivative ($p = q = 1$) should be weaker than the case of the second derivative ($p = q = 2$). If quadratic splines or other approximation methods were used, which assure the continuity of just the first derivative, then the results of using first- and second-order stabilizers would differ to a lesser degree.

Comparing the solutions as a function of the class of stabilizers introduced shows that the first class, $k = 1$, turns out to be best in the regularization scheme under consideration, i.e., a constraint on the domain of admissible solutions is realized best in the variable of the selected functional representation of the desired quantities with the variable domains of observation taken into account. The results obtained for the influence of the kind of stabilizing functional emphasize the value of using qualitative information about the smoothness of the desired quantities in solving inverse coefficient problems and show the inadequacy of constructing a stable solution requiring just matching the errors in the initial data.

The initial value $N = 5$ taken for the parameter turns out to be sufficient to determine the regularized solution in both the linear and nonlinear cases of the dependence of the thermophysical properties on the temperature. Therefore, utilization of the proposed regularization schemes in combination with the spline approximation permit restoration of both simple and complex dependences for sufficiently general assumptions about the desired quantities.

Let us examine how the passage to a total estimation of the measurement errors influences the accuracy of solving the inverse problem. For the case of nonlinear thermophysical properties and observations for $\sigma = 0.1$, utilization of the stabilizer $\Omega^{(1)}_{2,2}$ and a total estimation by means of the cubic norm $\Delta = \max_{i,j} |u^{\delta}_{ij} - \bar{u}_{ij}|$ would result in an increase in the error of the solution

TABLE 2. Identification of Nonlinear Laws of Thermophysical Property Variation with Estimation by the Cubic Norm (8)

Interference	$\max\limits_{i,j}\left\|\frac{u^{\delta}_{ij}-\bar{u}_{ij}}{\bar{u}_{ij}}\right\|$	Stabilizer	$\Omega^{(1)}_{0,0}$	$\Omega^{(1)}_{1,1}$	$\Omega^{(1)}_{2,2}$	$\Omega^{(1)}_{3,3}$	$\Omega^{(2)}_{1,1}$	$\Omega^{(2)}_{2,2}$	$\Omega^{(3)}_{1,1}$	$\Omega^{(3)}_{2,2}$	$\Omega^{(3)}_{3,3}$
$\sigma=0,01$	1,3%	$\int_{u_0}^{u_N}(a_1^{\delta}-\bar{a}_1)^2\,du$	0,00248	0,00248	0,00277	0,00265	0,00241	0,00272	0,00249	0,00281	0,00273
		$\int_{u_0}^{u_N}(a_2^{\delta}-\bar{a}_2)^2\,du$	0,01926	0,01933	0,01923	0,01966	0,01927	0,01924	0,01949	0,01939	0,01952
$\sigma=0,05$	5,1%	$\int_{u_0}^{u_N}(a_1^{\delta}-\bar{a}_1)^2\,du$	0,54951	0,10718	0,00536	0,01364	0,03097	0,01275	0,30794	0,17426	0,01374
		$\int_{u_0}^{u_N}(a_2^{\delta}-\bar{a}_2)^2\,du$	0,41255	0,33741	0,08369	0,06127	25,6591	0,23134	0,51674	0,47748	0,06135
$\sigma=0,1$	10,3%	$\int_{u_0}^{u_N}(a_1^{\delta}-\bar{a}_1)^2\,du$	0,69563	0,19471	0,00572	0,01435	0,08492	0,01953	0,35023	0,24301	0,01375
		$\int_{u_0}^{u_N}(a_2^{\delta}-\bar{a}_2)^2\,du$	0,83274	0,51841	0,42038	0,06372	31,5346	0,98113	0,61832	0,58633	0,06149

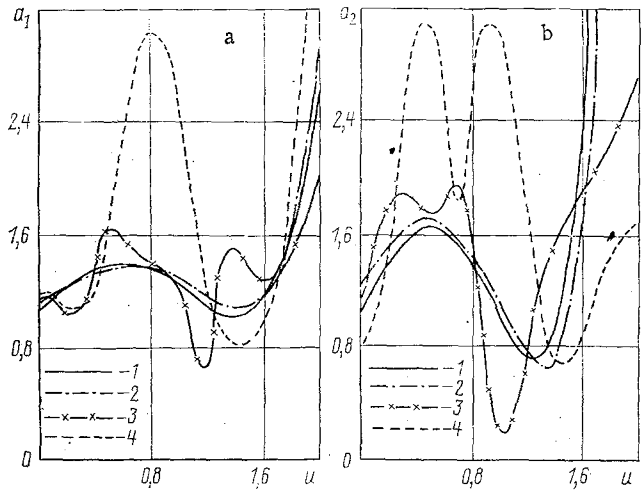

Fig. 2. Determination of nonlinear laws of specific heat (a) and heat conductivity (b) variation for $\sigma = 0.05$ and matching by the cubic norm (8): 1) exact values; 2) $\Omega^{(1)}_{2,2}$; 3) $\Omega^{(1)}_{1,1}$; 4) $\Omega^{(1)}_{0,0}$.

$\int_{u_0}^{u_N}(a_1^\delta - \bar{a}_1)^2 du = 3.4205$, $\int_{u_0}^{u_N}(a_1^\delta - \bar{a}_2)^2 du = 0.8004$. Estimation by the total rms error $\delta = \sum_i \sum_j (u_{ij}^\delta - \bar{u}_{ij})^2$ would result in still greater errors.

Therefore, the following deductions can be made. Selection of the method and the mode of the matching conditions is important in matching the inverse problem solution to the measurement error, i.e., the partial or total mode of taking account of the variance in the interference, the rms or uniform norm of the estimate of the deviations. However, just matching the measurement errors is insufficient for a satisfactory solution of the inverse problem and constraints must be imposed on the domain of admissible solutions. Weak regularization can result in a loss in the accuracy of identification despite compliance with the matching conditions. The regularization schemes studied with partial matching in the elements of a set of observations permit obtaining solutions of the inverse problems with satisfactory properties for an increase in the interference variance, where the errors of the solutions do not exceed the errors of the initial data.

In conclusion, we note that the investigation performed shows the effectiveness of regulaarization by the scheme of partial matching, which can be utilized for the practical solution of many inverse problems with the use of the broadest class of mathematical models.

NOTATION

$u(x, t)$, temperature field; $\bar{a}_{1,2}$, desired quantities; $a_{1,2}^\delta$, values found; $f(x, t)$, source intensity; Q, domain of variation of the independent variables; u^δ, set of observations; ε, interference in the measurements; x_i, points of observation; m, number of points of observation; t_j, times of the observations; n, number of times of observation; δ_i, rms deviation; Δ_i, maximal deviation; r_i residuals according to the matching conditions; and σ^2, interference variance.

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